Assignment 5.

This homework is due *Thursday*, October 2.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

1. Quick reminder

The (Lebesgue) outer measure is a function $m^*: \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined as

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; I_1, I_2, \dots \text{ open bounded intervals}, A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The outer measure m^* has the following properties:

- $m^*(I) = \ell(I)$ for every interval I.
- m^* is monotone: $m^*(A) \leq m^*(B)$ if $A \subseteq B$.
- m^* is translation invariant: for any $A \subseteq \mathbb{R}$, for any $y \in \mathbb{R}$,
$$m^*(A + y) = m^*(A).$$
- m^* is countably subadditive:
$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

2. Exercises

(1) (2.2.8+) Let B be the set of rational numbers in the interval $[0, 1]$. Let $\ell(I)$ denote length of an interval I.
   (a) Let $\{I_k\}$, $k = 1, 2, \dots, n$, be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^{n} \ell(I_k) \geq 1$.
   (b) Show that for every $\varepsilon > 0$, there is a countable open cover $\{I_k\}$, $k = 1, 2, \dots$, of B such that $\sum_{k=1}^{\infty} \ell(I_k) \leq \varepsilon$.

   Comment. This exercise shows that infinite covers may be very different from finite ones.
   Comment. Also, 1a is enough to do the exercise about finite covers given in class.

(2) (2.2.9) Prove directly from definition of m^* that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

(3) (2.2.10) Suppose A and B are *separated* bounded sets, i.e. there is an $\alpha \in \mathbb{R}$, $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove directly from definition of m^* that $m^*(A \cup B) = m^*(A) + m^*(B)$.

(4) Prove that a countable union of set of (Lebesgue) measure 0 is a set of measure 0:
   (a) directly from definition of m^*,
   (b) using subadditivity of m^*.

— see next page —

(5) (a) Define Jordan outer measure as a function $j^* : \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$j^*(A) = \inf \left\{ \sum_{k=1}^{n} \ell(I_k) \;\middle|\; I_1, \dots, I_n \text{ open intervals, } A \subseteq \bigcup_{k=1}^{n} I_k \right\}.$$

(That is, the same definition as m^*, except that only finite open covers are allowed.)
Prove that $j^*(\emptyset) = 0$, j^* is monotone, *finitely subadditive*, but *not countably* subadditive. (*Hint:* For the last part, see problem 1a.)

(b) Define function $c^* : \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$c^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; I_1, I_2, \dots \text{ closed bounded intervals, } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

(That is, the same definition as m^*, except that closed intervals covers are used instead of open ones.)
Prove that $c^* = m^*$.
(*Hint:* Prove that every closed intervals cover can itself be covered by open intervals with arbitrarily small "surplus", and vice versa.)

3. Extra exercise

(6) Prove that the Lebesgue outer measure m^* is not *continuous*, i.e. that it is not always true that

$$m^*\left(\bigcup_{k=1}^{\infty} A_k \right) = \lim_{n \to \infty} m^*\left(\bigcup_{k=1}^{n} A_k \right).$$